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ON CONE ORDERINGS AND THE LINEAR COMPLEMENTARITY PROBLEM. (U)

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ON CONE ORDERINGS AND THE LINEAR  
COMPLEMENTARITY PROBLEM

Jong-Shi Pang

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UNIVERSITY OF WISCONSIN - MADISON  
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ABSTRACT

This paper first generalizes a characterization of polyhedral sets having least elements, which is obtained by Cottle and Veinott [6], to the situation which Euclidean space is partially ordered by some general cone ordering (rather than the usual ordering). We then use this generalization to establish the following characterization of the class  $\mathcal{C}$  of matrices ( $\mathcal{C}$  arises as a generalization of the class of Z-matrices, see [4], [13], [14]):  $M \in \mathcal{C}$  if and only if for every vector  $q$  for which the linear complementarity problem  $(q, M)$  is feasible, the problem  $(q, M)$  has a solution which is the least element of the feasible set of  $(q, M)$  with respect to a cone ordering induced by some simplicial cone. This latter result generalizes the characterizations of K- and Z-matrices obtained by Cottle and Veinott [6] and Tamir [21] respectively.

AMS(MOS) Subject Classification - 90C99, 06A10, 15A39, 15A48

Key Words -- Cone orderings, least elements, linear complementarity, generalized Leontief.

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### EXPLANATION

The linear complementarity problem is nowadays an extremely important subject in mathematical programming. Its wide applications can be found in such diverse areas as optimization, economics, structural engineering, finance, free boundary value problems and optimal stopping. The area of large-scale linear complementarity problems has in recent years received an increasing amount of interest because of its potential usefulness in solving discretized partial differential equations. A fundamental problem arising here is the solution of these large-scale problems. Many of the currently available linear complementarity algorithms are either not applicable or not suitable to solve large-scale problems because of limited computer storage and expenses. Recently, Professor Mangasarian has proposed solving a class of large-scale linear complementarity problems as linear programs. The purpose of this paper is to provide a characterization of this class of problems in terms of some geometric notions in Euclidean space.

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ON CONE ORDERINGS AND THE LINEAR  
COMPLEMENTARITY PROBLEM

Jong-Shi Pang

1. INTRODUCTION

Recently, there are a number of papers in the literature which are concerned with characterizing polyhedral sets having least elements [6], with characterizing certain classes of matrices in terms of linear complementarity problems having least-element solutions [6], [16], [21], and with solving linear complementarity problems as linear programs [4], [5], [13], [14], [15], [18]. In fact, these three subjects are very closely related to each other. Among those papers mentioned above, the first one [6] seems to be the prime motivation for investigating the various relationships between the three theories. The essential result obtained in that paper is a theorem which characterizes polyhedral sets having least elements with respect to the usual ordering of Euclidean space. As an application of this characterization, the authors of the paper derived a characterization of the class of K-matrices in terms of linear complementarity problems having least-element solutions. It thus follows that linear complementarity problems with K-matrices can be solved as linear programs<sup>\*</sup>. The characterization of K-matrices was later extended to Z-matrices by Tamir [21]. Therefore, linear complementarity problems with Z-matrices can also be solved as linear programs.

An application exploiting the fact that linear complementarity problems with Z-matrices have least-element solutions has been described in [16]. See also [17]. In [16], the author studied a class of large-scale linear complementarity problems arising from quadratic programs with upper and lower bounds on the variables and with no other constraints. These quadratic programs have many applications in various areas. See [3]. We [16] presented a fast and efficient algorithm for this class of linear complementarity

<sup>\*</sup> See [4] for a more precise description of this concept. Throughout the paper, the phrase "solving linear complementarity problems as linear programs" has the meaning as described in [4], [13], [14].

problems which makes use of an efficient algorithm [2] to find the least-element solution of a linear complementarity problem with a Z-matrix.

Mangasarian, in two recent papers [13], [14], introduced certain new classes of matrices for which he showed that the corresponding linear complementarity problems can be solved as linear programs. His method of derivation has nothing to do with least elements. In [14], R. W. Cottle and the author summarized the results in Mangasarian's two papers by showing that all the classes of matrices studied in the paper - that are seemingly quite different - are in fact, subclasses of a large class of matrices, which we have denoted by  $\mathcal{C}$ . (Incidentally,  $\mathcal{C}$  includes the classes of Z- and K-matrices.) Moreover, we have shown that Mangasarian's results can indeed be derived via Cottle and Veinott's theory of polyhedral sets having least elements, thereby tightening up the connection between this latter theory and that of solving linear complementarity problems as linear programs.

It is natural to ask whether matrices in  $\mathcal{C}$  can in fact be characterized in terms of linear complementarity problems having least-element solutions. That this might be possible is suggested by the fact that it is possible for the classes of Z- and K-matrices. Our purpose in this paper is to provide a positive answer to this question. We would like to mention that a generalization of  $\mathcal{C}$  has been studied in [5] (least-element aspect) and in [15] (non-least-element aspect). Moreover, it has been shown [18] that  $\mathcal{C}$  is closely related to the well-known class  $\mathcal{K}$  of matrices whose characterization has long been an open problem in the theory of the linear complementarity problem but has recently been established in [18].

In order to characterize  $\mathcal{C}$ , we need to consider Euclidean space as being partially ordered by some general partial orderings (rather than just the usual ordering that is always implicitly implied in all the known characterizations). These partial orderings are induced by pointed cones and are thus called cone orderings. They certainly include the usual ordering as a special case. We shall develop a theory of polyhedral sets having elements that are least with respect to these cone orderings. The theory is an extension of that obtained in [6]. The key characterization theorem is described in terms of a generalized Leontief property presented in Saigal [19] and will be used to characterize  $\mathcal{C}$  in the manner described in the last paragraph. As in [6], we do not address the question of the existence of the least elements. Instead, we refer the interested reader to [17].

The plan of this paper is the following. The next section is a summary of background materials. It contains two parts. In the first part, we review some basic definitions and fix our notations. In the second part, we state some known results that are important to the development of our theory. In the third section, we consider  $R^n$  as being partially ordered by some cone ordering and develop a theory of polyhedral sets having least elements. We include a theorem characterizing a (strictly) isotone linear function on  $R^n$  under a cone ordering. This theorem is believed to be new and is related to the problem of finding the least element, provided that it exists. In the fourth and last section, we establish the promised characterization of  $C$ .



## 2. BACKGROUND

2.1. Basic Definitions and Notations. Throughout this paper,  $R_+^n$  will denote the non-negative orthant of Euclidean  $n$ -space  $R^n$ , and  $R^{n \times \ell}$  will denote the class of real  $n \times \ell$  matrices. We denote the  $i$ -th column (row) of a matrix  $A \in R^{n \times \ell}$  by  $A_{\cdot i}$  ( $A_{i \cdot}$ ). By  $e^i$  we denote the  $i$ -th unit vector, i.e. the vector whose components are all zero except the  $i$ -th component which is one. If  $S \subseteq R^n$ , we denote the interior of  $S$  by  $\text{int}(S)$ .

For a given vector  $q \in R^n$  and matrix  $M \in R^{n \times n}$ , we denote the linear complementarity problem of finding a vector  $x \in R_+^n$  satisfying

$$q + Mx \geq 0 \quad \text{and} \quad x^T(q + Mx) = 0$$

by the pair  $(q, M)$ . Note that the nonnegativity of the vectors  $x$  and  $(q + Mx)$  are meant componentwise. By the feasible set for the problem  $(q, M)$ , we mean the polyhedral set

$$X(q, M) = \{x \in R_+^n : q + Mx \geq 0\}.$$

The problem  $(q, M)$  is said to be feasible if  $X(q, M) \neq \emptyset$ .

A matrix  $M \in R^{n \times n}$  is said to be a Z-matrix if it has nonpositive off-diagonal entries. The Z-matrix  $M \in R^{n \times n}$  is said to be a K-matrix if there is a vector  $x \in R_+^n$  such that  $Mx > 0$  (componentwise). Properties of these two classes of matrices have been surveyed in [8]. Let  $A \in R^{n \times \ell}$ . It is said to be Leontief if it has at most one positive element in each column and there is a vector  $x \in R_+^\ell$  such that  $Ax > 0$  (componentwise). If  $A$  is Leontief, then the system

$$Ax = b \geq 0, \quad x \geq 0$$

is called a Leontief substitution system. Properties and applications of Leontief matrices and of the associated Leontief substitution systems are well recognized in the literature. See [7], [22]. It is clear that K-matrices are Leontief; and conversely, square Leontief matrices with positive diagonal elements are K-matrices. For  $A \in R^{n \times \ell}$  with full row rank, we say an  $n \times n$  submatrix  $B$  is a basis if it is nonsingular.

A partial ordering  $\leq$  (see [1]) on a set  $S$  is a binary relation on  $S$ , which satisfies for all  $x, y$  and  $z \in S$ , the following three axioms:

(P1)  $x \leq x$  (reflexivity);



(P2)  $x \leq y$  and  $y \leq z$  imply  $x \leq z$  (transitivity);

(P3)  $x \leq y$  and  $y \leq x$  imply  $x = y$  (antisymmetry).

The set  $S$  is said to be partially ordered (by  $\leq$ ) or a poset if  $\leq$  is a partial ordering on  $S$ . We denote the poset  $S$  together with the partial ordering  $\leq$  by the pair  $(S, \leq)$ . It is clear that every subset of a poset is a poset with the induced ordering. If  $x$  and  $y$  are elements in a poset  $S$ ,  $x \leq y$  and  $x \neq y$ , then we write  $x < y$ .

Example. The usual ordering  $\leq$  of  $R^n$  is defined as follows: For  $x, y \in R^n$ ,  $x \leq y$  if and only if  $x_i \leq y_i$  for every  $i$ . It is trivial to show that this is a partial ordering. Later in this section, the usual ordering of  $R^n$  will be generalized. For this particular ordering, we write  $x < y$  to mean  $x_i < y_i$  for every  $i$ . This is not to be confused with  $<$  which is used for other orderings and has a weaker meaning.

Let  $T$  be a subset of the poset  $(S, \leq)$ . An element  $t^* \in T$  is a least element of  $T$  (with respect to or under  $\leq$ ) if  $t^* \leq t$  for every  $t \in T$ . The least element of a poset, if it exists, must be unique. This follows immediately from its definition and the antisymmetry of the partial ordering.

Let  $(S, \leq)$  and  $(S', \leq')$  be posets. A mapping  $f: S \rightarrow S'$  is said to be isotone if  $x, y \in S$  and  $x \leq y$  imply  $f(x) \leq' f(y)$ . An isotone mapping is strict if  $x < y$  implies  $f(x) <' f(y)$ . When  $S'$  is the real line and  $\leq$  is the usual ordering  $\leq$  of scalars, we say that  $f$  is a (strictly) isotone real-valued function if the mapping  $f: (S, \leq) \rightarrow (R, \leq)$  is (strictly) isotone.

We review a few concepts about cones in Euclidean space [20]. A subset  $C$  of  $R^n$  is called a cone if it satisfies the following three conditions:

(C1)  $0 \in C$ ;

(C2)  $\lambda x \in C$  for every  $\lambda \in R_+$  and  $x \in C$ ;

(C3)  $x + y \in C$  for  $x, y \in C$ .

The cone  $C$  is said to be pointed if  $x \in C$  and  $-x \in C$  imply  $x = 0$ . It is finitely generated if there exists  $G \in R^{n \times m}$  such that  $C = \{x \in R^n: x = Gy \text{ for some } y \in R_+^m\}$ .

In this case, we denote  $C$  by  $\text{pos}(G)$ . A cone  $C$  is polyhedral if there exists  $F \in R^{m \times n}$  such that  $C = \{x \in R^n: Fx \geq 0\}$ . A cone  $C$  is simplicial if  $C = \text{pos}(X)$  for some  $X \in R^{n \times n}$  and  $X$  is nonsingular.

Note that conditions (C2) and (C3) together imply that cones are convex. It is well-known (see [20]) that a cone is finitely generated if and only if it is polyhedral. For an arbitrary subset  $S \subseteq \mathbb{R}^n$ , we define its polar cone

$$S^* = \{y \in \mathbb{R}^n: x^T y \geq 0 \text{ for every } x \in S\}.$$

Obviously,  $S^*$  is a nonempty closed cone. The next proposition can easily be proved (see [20]). It gives an explicit formula for the polar cone of a finitely generated cone.

Proposition 2.1. Let  $G \in \mathbb{R}^{n \times m}$ . Then

$$\text{pos}(G)^* = \{x \in \mathbb{R}^n: x^T G \geq 0\}.$$

The following observation is important. Namely, if the matrix  $G \in \mathbb{R}^{n \times m}$  has linearly independent columns, then the cone  $\text{pos}(G)$  is pointed and its polar cone  $\text{pos}(G)^*$  must have a nonempty interior. This latter property is an immediate consequence of a standard alternative theorem on the solvability of a homogeneous system of linear equations. See [20] e.g. In particular, a simplicial cone is pointed and its polar cone has a nonempty interior.

We now return to orderings. The usual ordering  $\leq$  of  $\mathbb{R}^n$  is defined "component-wise" and determines a "cone of nonnegativity"  $C = \{x \in \mathbb{R}^n: 0 \leq x\}$  which, in this case, is precisely the nonnegative orthant  $\mathbb{R}_+^n$ . This ordering is generalized in the following manner. (See Kransnoselski [12].) Let  $C$  be an arbitrary pointed cone in  $\mathbb{R}^n$ . The cone ordering  $\leq_C$  (induced by the pointed cone  $C$ ) is defined as follows: for  $x, y \in \mathbb{R}^n$ ,  $x \leq_C y$  if and only if  $y - x \in C$ . It can readily be verified that this is a partial ordering. Indeed, axiom (P1) follows from (C1), (P2) from (C3) and (P3) from the pointedness of  $C$ . Under this cone ordering  $\leq_C$ ,  $C$  becomes the cone of nonnegativity, i.e.  $C = \{x \in \mathbb{R}^n: 0 \leq_C x\}$ . It is clear that the cone ordering induced by the nonnegative orthant is precisely the usual ordering. If  $C$  is the polyhedral cone  $\{x \in \mathbb{R}^n: Fx \geq 0\}$  where  $F \in \mathbb{R}^{m \times n}$  and the columns of  $F$  are linearly independent (in fact, the linear independence is equivalent to the pointedness of  $C$ ), then the cone ordering  $\leq_C$  can be viewed as a replica of the usual ordering as applied to a transformed image of  $\mathbb{R}^n$ . Indeed, consider the linear transformation  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by  $L(x) = Fx$ . Then the

ordering  $\leq_C$  is precisely the "inverse image" of the usual ordering on the image space  $L(\mathbb{R}^n)$ ; in other words,  $x \leq_C y$  if and only if  $L(x) \leq L(y)$ .

The idea of incorporating cone orderings in the study of complementarity problems is no new subject at all. In fact, the generalized complementarity problem which was introduced by Habetler and Price [9] and later refined by Karamardian [10] is defined by means of a cone ordering. We shall not discuss this latter problem further but refer the interested reader to the paper [10].

**2.2. Known Results.** In this subsection, we review some known results that are of fundamental importance in this paper. In each of the four theorems stated below, the word "least" is meant "least under the usual ordering", i.e. "least componentwise". The first theorem characterizes polyhedral sets having least elements.

**Theorem 2.2.** (Cottle and Veinott [6]) Let  $A \in \mathbb{R}^{\ell \times n}$  and  $X_b = \{x \in \mathbb{R}^n : Ax \geq b\}$  for  $b \in \mathbb{R}^\ell$ . The following are equivalent:

- (2.1a)  $X_b$  has a least element for each  $b$  such that  $X_b$  is nonempty.
- (2.1b) There is a basis  $B$  in  $A^T$  for which  $B^{-1}c \geq 0$  for some  $c > 0$  and each such basis has a nonnegative inverse (entrywise).

It was noted in [6] that  $X_b$  is nonempty for all  $b$  if and only if there is an  $x$  such that  $Ax > 0$ . Applying Theorem 2.2 to the linear complementarity problem  $(q, M)$ , we obtain the following characterization of K-matrices.

**Theorem 2.3.** (Cottle and Veinott [6]) Let  $M \in \mathbb{R}^{n \times n}$ . The following are equivalent:

- (2.2a)  $M$  is a K-matrix.
- (2.2b) For each  $q \in \mathbb{R}^n$ ,  $X(q, M)$  has a least element  $\bar{x}$  and  $\bar{x}$  is the unique element in  $X(q, M)$  satisfying  $\bar{x}^T(q + M\bar{x}) = 0$ .

A similar characterization of Z-matrices is given by

**Theorem 2.4.** (Tamir [21]) Let  $M \in \mathbb{R}^{n \times n}$ . The following are equivalent:

- (2.3a)  $M$  is a Z-matrix.
- (2.3b) For each  $q \in \mathbb{R}^n$  such that  $X(q, M) \neq \emptyset$ ,  $X(q, M)$  has a least element  $\bar{x}$  and  $\bar{x}$  satisfies  $\bar{x}^T(q + M\bar{x}) = 0$ .



The phrase "linear complementarity problems having least-element solutions" used in the introduction, originates from these last two theorems. It is an abbreviation for "linear complementarity problems which have solutions that can be characterized as the least elements of the feasible sets of the problems."

The class  $\mathcal{C}$  of matrices consists of those real  $n \times n$  matrices  $M$  which, together with  $Z$ -matrices  $X$  and  $Y$ , satisfy the following two conditions:

$$(M1) \quad MX = Y$$

$$(M2) \quad r^T X + s^T Y > 0 \text{ for some } r, s \in \mathbb{R}_+^n.$$

Proposition 2.5. (Cottle and Pang [4]) Let  $M \in \mathcal{C}$ . Let  $X$  and  $Y$  be  $Z$ -matrices satisfying conditions (M1) and (M2). Then

$$(2.4a) \quad X \text{ is nonsingular}$$

$$(2.4b) \quad (X^T, Y^T) \text{ is a Leontief matrix.}$$

Theorem 2.6. (Cottle and Pang [4]) Let  $M$ ,  $X$  and  $Y$  satisfy the assumptions in Proposition 2.5. Suppose that the linear complementarity problem  $(q, M)$  is feasible. Then the polyhedral set

$$(2.5) \quad V_q = \{v \in \mathbb{R}^n : Xv \geq 0, q + Yv \geq 0\}$$

has a least element  $\bar{v}$ . Furthermore the vector  $\bar{x} = X\bar{v}$  solves the problem  $(q, M)$  and  $\bar{x}$  can be obtained by solving the linear program

$$(2.6) \quad \text{minimize } p^T x \text{ subject to } x \in X(q, M)$$

where  $p$  satisfies  $p^T X > 0$ .



### 3. POLYHEDRAL SETS HAVING LEAST ELEMENTS

From now on, we consider  $R^n$  as being partially ordered by some general partial ordering. We first state a proposition which shows how the least element of a set  $S \subseteq R^n$  can be obtained, provided that it exists. The proof of the proposition is straightforward and thus omitted.

Proposition 3.1. Let  $\leq$  be a partial ordering on  $R^n$ . Let  $\bar{x}$  be the least element of a set  $S \subseteq (R^n, \leq)$ . Then  $\bar{x}$  is a solution to the minimization problem

$$(3.1) \quad \text{minimize } f(x) \text{ subject to } x \in S$$

for every isotone real-valued function  $f$  defined on  $S$ . Furthermore,  $\bar{x}$  is the unique solution if  $f$  is strictly isotone.

The significance of the proposition is that it suggests a constructive (though sometimes not too effective) approach to find the least element whenever it exists. If it happens that  $S$  is a polyhedral set in  $R^n$ , and if  $f$  is chosen to be linear, then problem (3.1) reduces to a linear program which of course, can be solved by the simplex method of linear programming.

In what follows, we present a representation theorem for linear (strictly) isotone real-valued function defined on  $(R^n, \leq_C)$  where  $C$  is some finitely generated pointed cone.

Theorem 3.2. Let  $G \in R^{n \times m}$  have linearly independent columns and let  $C = \text{pos}(G)$ . Consider the poset  $(R^n, \leq_C)$ . A linear function  $f: R^n \rightarrow R$  is (strictly) isotone if and only if there exists a vector  $p \in (\text{int } C^*) \cap C^*$  such that  $f(x) = p^T x$  for every  $x \in R^n$ .

Proof: We prove only the case of strict isotonicity. Proposition 2.1 implies that

$$\text{int } C^* = \{x \in R^n: x^T G > 0\}.$$

If  $f(x) = p^T x$  for some  $p \in \text{int } C^*$  and for every  $x \in R^n$ , then

$$x \leq_C y \Rightarrow y - x = Gr \text{ for some } r \in R_+^m \setminus \{0\}$$

$$\Rightarrow f(y) - f(x) = p^T(y - x) = (p^T G)r > 0.$$

Thus,

$$x <_C y \Rightarrow f(x) < f(y) .$$

Hence  $f$  is strictly isotone. Conversely, suppose that  $f$  is strictly isotone. The linearity of  $f$  implies the existence of a vector  $p \in R^n$  such that  $f(x) = p^T x$  for every  $x \in R^n$ . It remains to verify  $p \in \text{int } C^*$ . Let  $r \in R_+^m \setminus \{0\}$ . Then  $0 <_C Gr$  because  $G$  has linearly independent columns. The strict isotonicity of  $f$  therefore implies that  $f(Gr) = (p^T G)r > 0$ . In other words, we have proved the implication

$$r \in R_+^m \setminus \{0\} \Rightarrow (p^T G)r > 0$$

which is clearly equivalent to  $p^T G > 0$ . Therefore  $p \in \text{int } C^*$ . This completes the proof of the theorem.

Combining Theorem 3.2 with Proposition 3.1, we obtain

Corollary 3.3. Let  $C$  be given in Theorem 3.2. Let  $\bar{x}$  be the least element of  $S \subseteq (R^n, \leq_C)$ . Then  $\bar{x}$  is a solution to the minimization problem

$$\text{minimize } p^T x \text{ subject to } x \in S$$

for every  $p \in C^*$ . Furthermore, it is the unique solution if  $p \in \text{int } C^*$ .

Remark. In both Theorem 3.2 and Corollary 3.3, the cone  $C$  is not required to be simplicial.

The conclusion in Theorem 2.6 about how the vector  $\bar{x}$  can be obtained is an immediate consequence of Corollary 3.3 because as we shall see later,  $\bar{x}$  is indeed the least element of the feasible set  $X(q, M)$  under the cone order  $\leq_{\text{pos}(X)}$ .

In order to characterize polyhedral sets having least elements with respect to cone orderings, we introduce the following two definitions.

Definition 3.4. (Saigal [19]) Let  $C$  be a convex set in  $R^n$  with  $\text{int } C \neq \emptyset$ . Let  $A \in R^{n \times \ell}$  with full row rank. We say that  $A$  has the generalized Leontief property with respect to  $C$  if the following two conditions are satisfied:

(3.2a) there is a basis  $B$  of  $A$  such that  $C \subseteq \text{pos}(B)$  ;

(3.2b) for each basis  $B$  of  $A$  such that  $\text{int } C \cap \text{pos}(B) \neq \emptyset$ , we have  $C \subseteq \text{pos}(B)$  .

We define  $L(C)$  to be the set of all matrices which have the generalized Leontief property with respect to  $C$ .

**Definition 3.5.** (Saigal [19]) We say that a matrix  $A \in L(C)$  is hidden Leontief if there is a nonsingular matrix  $D$  such that  $DA$  is Leontief and  $C \subseteq \text{pos}(D^{-1})$ .

Motivation to study the generalized Leontief property and hidden Leontief matrices is due to the fact that "there are constraint sets  $\{x \in R_+^n: Ax = b\}$  that do not appear to be Leontief substitution systems but can be shown to be equivalent to such systems". Various characterizations of hidden Leontief matrices have been obtained by Saigal [19]. See also [11]. An application which exploits hidden Leontief properties is given in [11]. If  $A$  is hidden Leontief, then the system

$$Ax = b, \quad x \geq 0$$

is called a hidden Leontief substitution system.

**Theorem 3.6.** Let  $C$  be a simplicial cone in  $R^n$ . Consider  $(R^n, \leq_C)$ . Let  $A \in R^{\ell \times n}$  and  $X_b = \{x \in R^n: Ax \geq b\}$  for  $b \in R^\ell$ . Then the following are equivalent:

(3.3a)  $X_b$  has a least element for each  $b$  such that  $X_b \neq \emptyset$ .

(3.3b)  $A^T \in L(C^*)$ .

**Proof:** For each  $b \in R^\ell$ , let

$$Y_b = \{y \in R^n: Axy \geq b\}$$

where  $C = \text{pos}(X)$  with  $X \in R^{n \times n}$  and nonsingular. It is clear that  $X_b$  is nonempty if and only if  $Y_b$  is so; and  $\bar{x}$  is the least element of  $X_b$  with respect to  $\leq_C$  if and only if  $\bar{y} = X^{-1}\bar{x}$  is the least element of  $Y_b$  with respect to the usual ordering. Hence, (3.3a) is equivalent to

(3.4) for each  $b$  such that  $Y_b \neq \emptyset$ ,  $Y_b$  has a least element with respect to the usual ordering.

According to Theorem 2.2, (3.4) is equivalent to

(3.5) there is a basis  $B'$  in  $(AX)^T$  for which  $(B')^{-1}c \geq 0$  for some  $c > 0$  and each such basis  $B'$  has a nonnegative inverse.

Every basis  $B'$  of  $(AX)^T$  has the form  $B' = (BX)^T$  where  $B^T$  is a basis of  $A^T$ . The converse is also true, i.e., if  $B^T$  is a basis of  $A^T$ , then  $B' = (BX)^T$  is a basis of  $(AX)^T$ . Therefore, (3.5) is equivalent to

(3.6) there is a basis  $B^T$  of  $A^T$  for which  $(B^T d)^T x > 0$  for some  $d \geq 0$  and each such basis  $B^T$  satisfies  $[c \geq 0 \Rightarrow (BX)^{-T} c \geq 0]$ .

Noting that

$$(BX)^{-T} c \geq 0 \Leftrightarrow x^{-T} c = B^T d \text{ for some } d \in \mathbb{R}_+^n$$

and that  $C^* = \text{pos}(x^{-T})$ , we conclude readily that (3.6) is indeed equivalent to (3.3b). This completes the proof of the theorem.

Theorem 3.6 characterizes polyhedral sets having least elements with respect to partial orderings induced by simplicial cones. It generalizes Theorem 2.2. It should be pointed out that the requirement that the cone  $C$  be simplicial is essential in order for the one-to-one correspondence between elements in  $X_b$  and  $Y_b$  and also for the relationship between basis of  $(AX)^T$  and of  $A^T$  to be valid. If  $C$  is merely finitely generated and pointed, we have the following result.

Proposition 3.7. Let  $G \in \mathbb{R}^{n \times m}$  have linearly independent columns and  $C = \text{pos}(G)$ . Consider  $(\mathbb{R}^n, \leq_C)$ . Let  $A \in \mathbb{R}^{l \times n}$ . If  $A^T \in L(C^*)$  and is hidden Leontief, then (3.3a) holds.

Proof: According to a property of hidden Leontief matrices [19], there exists a simplicial cone  $S = \{x \in \mathbb{R}^n \mid Dx \geq 0\}$  such that  $A^T \in L(S)$  and  $C^* \subseteq S$ . Since  $S = (S^*)^*$  (see [20]), it follows from Theorem 3.6 that (3.3a) holds with respect to  $S^*$ . Therefore, (3.3a) must hold with respect to  $C$  because  $S^* \subseteq (C^*)^* = C$ . This completes the proof of the proposition.



#### 4. CHARACTERIZATION OF $C$

In this section, we use the results developed in the last section to establish a characterization of  $C$  in terms of linear complementarity problems having least-element solutions. Before proving the main theorem, we state and prove the following proposition which describes a relationship between matrices in  $C$  and hidden Leontief matrices. The proposition generalizes the fact that if  $M$  is a Z-matrix, then the matrix  $(I, M^T)$  is Leontief.

**Proposition 4.1.** Let  $M \in R^{n \times n} \cap C$ . Then there exists a simplicial cone  $C^*$  such that the matrix  $A^T = (I, M^T) \in L(C^*)$  and is hidden Leontief.

**Proof:** Let  $X$  and  $Y$  be Z-matrices satisfying conditions (M1) and (M2). Observe that (M2) can be written as

$$(r^T + s^T M)X > 0 \text{ for some } r, s \in R_+^n.$$

Let  $C = \text{pos}(X)$ . Then  $C^* = \text{pos}(X^{-T})$  is simplicial. According to Theorem 2.6, for every vector  $q$  such that  $X(q, M) \neq \emptyset$ , the polyhedral set  $V_q$  defined by (2.5) has a least element  $\bar{v}$  with respect to the usual ordering. As mentioned in Theorem 3.6,  $\bar{v}$  is the least element of  $V_q$  with respect to the usual ordering if and only if  $\bar{x} = X\bar{v}$  is the least element of  $X(q, M)$  with respect to the cone ordering  $\leq_C$ . Therefore by Theorem 3.6, we conclude that  $A^T \in L(C^*)$ . It remains to verify that  $A^T$  is hidden Leontief. Noting that  $(X^T, Y^T) = X^T A^T$  and letting  $D = X^T$ , we deduce, by (2.4b), that  $DA^T$  is Leontief. Finally, it is clear that  $C^* = \text{pos}(D^{-1})$ . This completes the proof of the proposition.

**Remark:** The matrix  $A^T = (I, M^T)$  arises in the linear programming formulation of the linear complementarity problem  $(q, M)$  with  $M \in C$ . Indeed, the dual of the linear program (2.6) is given by:

$$\text{minimize } q^T y \text{ subject to } p - M^T y \geq 0, y \geq 0$$

or equivalently,

$$\text{minimize } \begin{pmatrix} 0 \\ q \end{pmatrix}^T \begin{pmatrix} v \\ y \end{pmatrix} \text{ subject to } A^T \begin{pmatrix} v \\ y \end{pmatrix} = p, \begin{pmatrix} v \\ y \end{pmatrix} \geq 0.$$

Combining Proposition 4.1 and Theorem 2.6, we conclude that if  $M \in \mathcal{C}$  and if the linear complementarity problem  $(q, M)$  is feasible, then  $(q, M)$  has a solution which can be obtained by solving a linear program whose dual has a constraint set defined by a hidden Leontief substitution system.

**Theorem 4.2.** Let  $M \in \mathbb{R}^{n \times n}$ . The following are equivalent:

$$(4.1a) \quad M \in \mathcal{C}.$$

$$(4.1b) \quad \text{There exists a simplicial cone } C, \text{ such that for each } q \in \mathbb{R}^n \text{ for which } X(q, M) \neq \emptyset, X(q, M) \text{ has a least element } \bar{x} \text{ with respect to } \prec_C \text{ and } \bar{x} \text{ satisfies } \bar{x}^T (q + M\bar{x}) = 0.$$

**Proof:** (4.1a)  $\Rightarrow$  (4.1b). This follows immediately from Theorem 2.6 and the proof of Proposition 4.1.

(4.1b)  $\Rightarrow$  (4.1a). Let  $C = \text{pos}(X)$  where  $X \in \mathbb{R}^{n \times n}$  is nonsingular but not necessarily a Z-matrix. Let  $Y = MX$ . According to the assumption, we deduce that for every vector  $q$  for which the set  $V_q$  (defined by (2.5)) is nonempty, it has a least element  $\bar{v} (= X^{-1}x)$  with respect to the usual ordering and  $\bar{v}$  satisfies  $(X\bar{v})^T (q + Y\bar{v}) = 0$ .

Let  $k$  be an index in  $\{1, \dots, n\}$ . Choose  $q = e^k - Ya^k$  where  $a^k = X^{-1}e^k$ . Then  $a^k \in V_q$ . Therefore  $V_q$  contains an element  $\bar{v}^k \leq a^k$ . Moreover,  $\bar{v}^k \neq a^k$ . Define  $v^k = a^k - \bar{v}^k$ , then  $v^k$  is a non-vanishing vector. For  $i \neq k$ , we have

$$(4.2a) \quad x_{i \cdot} v^k = x_{i \cdot} (X^{-1}e^k) - x_{i \cdot} \bar{v}^k \leq 0$$

and

$$(4.2b) \quad y_{i \cdot} v^k = y_{i \cdot} a^k - y_{i \cdot} \bar{v}^k = q_i - y_{i \cdot} \bar{v}^k \leq 0.$$

Now if we define the matrix  $W = (v^1, \dots, v^n)$  where  $v^k$  are the vectors defined above, then clearly,  $X' = XW$  and  $Y' = YW$  are Z-matrices by (4.2). Moreover  $MX' = Y'$ . It remains to verify that there exist vectors  $r$  and  $s$  in  $\mathbb{R}_+^n$  such that  $r^T X' + s^T Y' > 0$ . By Theorem 3.6, it follows that  $A^T = (I, M^T) \in L(C^*)$ . Therefore, there exists a basis  $B^T$  of  $A^T$  such that  $C^* \subseteq \text{pos}(B^T)$ . This implies, by the fact that  $\text{int } C^* \neq \emptyset$ .

$$(t^T B)X > 0 \text{ for some } t \in \mathbb{R}_+^n.$$

If we define the vectors  $r = (r_i)$  and  $s = (s_i)$

$$r_i = \begin{cases} t_j & \text{if } e^i = (B^T)_{\cdot j} \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$

$$s_i = \begin{cases} t_j & \text{if } (M^T)_{\cdot i} = (B^T)_{\cdot j} \text{ for some } j \\ 0 & \text{otherwise,} \end{cases}$$

then clearly  $r, s \in R_+^n$  and

$$r^T X + s^T Y = (r^T + s^T M)X = (t^T B)X > 0.$$

Finally, we have

$$r^T X' + s^T Y' = (r^T X + s^T Y)W > 0$$

because each column of  $W$  is non-vanishing and  $W \geq 0$ . This shows that  $M \in \mathcal{C}$  and completes the proof of the theorem.

The above theorem generalizes both Theorems 2.3 and 2.4. It should be emphasized that in (4.1b), the cone  $\mathcal{C}$  is not required to be induced by a Z-matrix. Moreover, it is worth pointing out that there exist matrices  $M \in \mathcal{C}$  which satisfy the defining conditions (M1) and (M2) for some matrices  $X$  and  $Y$  which are not both Z-matrices. An example is given by the following.

Example. Consider  $M = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ . Let  $X = \begin{pmatrix} 1 & -1 \\ -1/2 & 1 \end{pmatrix}$  and  $Y = \begin{pmatrix} 2 & -1 \\ 1/2 & -1 \end{pmatrix}$ . Clearly  $MX = Y$  and (M2) is satisfied because  $X$  is a K-matrix.  $M \in \mathcal{C}$  because

$$\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix}.$$

Observe that  $Y$  is not a Z-matrix.

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#### REFERENCES

- [1] Birkhoff, G., Lattice theory, 3rd edition, American Mathematical Society, Providence, Rhode Island, 1967.
- [2] Chandrasekaran, R., "A special case of the complementary pivot problem", Opsearch 7 (1970), 263-268.
- [3] Cottle, R. W. and M. Goheen, "On a special class of large quadratic programs", Tech. Report SOL 76-7, Systems Optimization Laboratory, Department of Operations Research, Stanford University, 1976.
- [4] Cottle, R. W. and J. S. Pang, "On solving linear complementarity problems as linear programs", Tech. Report 76-5, Systems Optimization Laboratory, Department of Operations Research, Stanford University, 1976.
- [5] Cottle, R. W. and J. S. Pang, "A least-element theory of solving linear complementarity problems as linear programs", MRC Tech. Report #1702, Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, 1976.
- [6] Cottle, R. W. and A. F. Veinott, Jr., "Polyhedral sets having a least element", Mathematical Programming 3 (1972), 238-249.
- [7] Dantzig, G. B., "Optimal solution of a dynamic Leontief model with substitution", Econometrica 23 (1955), 295-302.
- [8] Fiedler, M. and V. Pták, "On matrices with non-positive off-diagonal elements and positive principal minors", Czech. Mathematics Journal 12 (1962), 382-400.
- [9] Habetler, G. J. and A. J. Price, "Existence theory for generalized nonlinear complementarity problems", J. Optimization Theory Appl. 7 (1971), 223-239.
- [10] Karamardian, S., "Generalized complementarity problem", J. Optimization Theory Appl. 8 (1971), 161-168.
- [11] Koehler, G. J., A. B. Whinston and G. P. Wright, Optimization over Leontief substitution systems, North-Holland Publishing Company, Amsterdam, The Netherlands 1975.
- [12] Kransnoselski, M. A., Positive solutions of operator equations, P. Noordhoff Ltd., Groningen, Holland 1964.



- [13] Mangasarian, O. L., "Linear complementarity problems solvable by a single linear program", Mathematical Programming 10 (1976), 263-270.
- [14] Mangasarian, O. L., "Solution of linear complementarity problems by linear programming", in Numerical Analysis, Dundee 1975, G. W. Watson (ed.), Lecture Notes in Mathematics, No. 506, Springer-Verlag, Berlin 1976, 166-175.
- [15] Mangasarian, O. L., "Characterization of linear complementarity problems as linear programs", Tech. Report 271, Computer Sciences Dept., University of Wisconsin, Madison, May 1976.
- [16] Pang, J. S., "On a class of least-element complementarity problems", Tech. Report SOL 76-10, Systems Optimization Laboratory, Department of Operations Research, Stanford University, 1976.
- [17] Pang, J. S., "Least element complementarity problems", Ph.D. dissertation, Department of Operations Research, Stanford University, September 1976.
- [18] Pang, J. S., "A note on an open problem in linear complementarity", submitted to Mathematical Programming.
- [19] Saigal, R., "On a generalization of Leontief substitution systems", working paper, Bell Telephone Laboratories, Holmdel, N.J.
- [20] Stoer, J. and C. Witzgall, Convexity and optimization in finite dimensions: I, Springer-Verlag, New York 1970.
- [21] Tamir, A., "The complementarity problem of mathematical programming", Ph.D. dissertation, Department of Operations Research, Case Western Reserve University, June 1973.
- [22] Veinott, A. F., Jr., "Extreme points of Leontief substitution systems", Linear Algebra and its Applications 1 (1968), 181-194.

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of  $(q, M)$  with respect to a cone ordering induced by some simplicial cone. This latter result generalizes the characterizations of  $K$ - and  $Z$ -matrices obtained by Cottle and Veinott [6] and Tamir [21] respectively.

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